

# Incremental and Decremental Secret Key Agreement

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**Abstract**—We study the rate of change of the multivariate mutual information among a set of random variables when some common randomness is added to or removed from a subset. This is formulated more precisely as two new multiterminal secret key agreement problems which ask how one can increase the secrecy capacity efficiently by adding common randomness to a small subset of users, and how one can simplify the source model by removing redundant common randomness that does not contribute to the secrecy capacity. The combinatorial structure has been clarified along with some meaningful open problems.

**Index Terms**—secret key agreement, multivariate mutual information, principal sequence of partitions.

## I. INTRODUCTION

We consider the multiterminal secret key agreement problem formulated by [1]. A group of users, each observing a private correlated random source, discuss in public so they can agree on a secret key. The key is a random variable that needs to be recoverable by every user after the discussion. Furthermore, the key must be secured against a wiretapper who can observe the entire discussion among the users but has no access to their private sources.

The maximum achievable secret key rate is called the *secrecy capacity*. It was characterized by [1] as a linear program. Because the wiretapper can listen to the entire public discussion, *the randomness of the secret key can only come from the information mutual to the private correlated source components*. Indeed, in the two-user case, it was shown in [1] that the capacity is equal to *Shannon's mutual information*:

$$I(Z_1 \wedge Z_2) = D(P_{Z_1 Z_2} \| P_{Z_1} P_{Z_2}), \quad (1.1)$$

where each user  $i \in \{1, 2\}$  observes the discrete memoryless correlated private source  $Z_i$ . The mutual information above is written as the *divergence*  $D$  from the joint distribution  $P_{Z_1 Z_2}$  to the product of the marginal distributions  $P_{Z_1}$  and  $P_{Z_2}$ .

In the multiterminal case, let  $V$  be the set of (two or more) users, and  $Z_V := (Z_i \mid i \in V)$  be a random vector where  $Z_i$  is a discrete memoryless source component privately observed by user  $i \in V$ . There was a divergence upper bound on the capacity in [1], which was identified [2, 3] to be tight in the special case without helpers, giving rise to the alternative

capacity characterization:

$$I(Z_V) := \min_{\mathcal{P} \in \Pi'(V)} I_{\mathcal{P}}(Z_V), \quad \text{where} \quad (1.2a)$$

$$I_{\mathcal{P}}(Z_V) := \frac{D(P_{Z_V} \| \prod_{C \in \mathcal{P}} P_{Z_C})}{|\mathcal{P}| - 1}, \quad (1.2b)$$

and  $\Pi'(V)$  is the collection of partitions  $\mathcal{P}$  of  $V$  into two or more non-empty disjoint sets.

Following [4], we call (1.2a) the *multivariate mutual information (MMI)*. It is easy to see that (1.1) is a special case of (1.2a) when  $V = \{1, 2\}$ . Indeed, the MMI was formally regarded in [4] as a measure of mutual information among multiple random variables, thereby extending various interpretations and properties of Shannon's mutual information to the multivariate case. The MMI has other operational meanings, ranging from tree-packing [5], hypergraph connectivity [3], undirected network coding [6], vocality constraints [7–9], successive omniscience [10, 11] and data clustering [12, 13].

In this work, we want to study how the MMI of a set of random variables changes with respect to changes in the MMI of a subset of the random variables. We formulate two new problems, called the *incremental secret key agreement (ISKA)* and *decremental secret key agreement (DSKA)*. In ISKA, a subset of users is given an additional common randomness in the form of a random source of certain entropy rate. The objective is to find an efficient resource allocation, i.e., to increase the capacity as much as possible without requiring too much common randomness to be added to too many users. In DSKA, we remove some common randomness from a subset of users. The objective is to simplify the source model, but without reducing the capacity much. In particular, we want to identify redundant common randomness whose removal does not diminish the capacity.<sup>1</sup>

## II. MOTIVATION

We first explain the idea using a simple example. Define the random source as

$$Z_1 := (X_a, X_b), \quad Z_2 := (X_a, X_b), \quad \text{and} \quad Z_3 := X_a,$$

where  $X_a$  and  $X_b$  are independent uniformly random bits. The random bits  $X_j$ 's determine the correlation, or joint distribution, of the sources  $Z_i$ 's. Let  $V := \{1, 2, 3\}$  be the set of users. Each user  $i \in V$  observes the discrete memoryless

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<sup>1</sup>The idea of redundant common randomness first appeared in [3, Proposition 2.3]. It was called excess edge under a hypergraphical source model, and was related to the notion of partition connectivity for hypergraphs. The idea was also used in [14] in characterizing the minimum communication complexity for secret key agreement, but under a non-asymptotic hypergraphical source model when the communication protocol is restricted to be linear. [15–17] further considered the non-asymptotic case and derived more efficiently computable bounds on the communication complexity.

source  $Z_i$  privately. It is easy to see that the users can agree on a secret key bit, namely,  $X_a$ , without any public discussion. In fact, the users cannot agree on any more secret key bits, even with additional public discussion. This is clear since  $X_a$  is already the entire private observation of user 3. *The secrecy capacity is therefore 1 bit.*

For ISKA, we consider adding a common randomness to a subset of users. For example, we may add a new independent bit  $X_c$  to the private sources of users 2 and 3 as follows:

$$Z'_1 := (X_a, X_b), \quad Z'_2 := (X_a, X_b, X_c), \quad \text{and} \quad Z'_3 := (X_a, X_c),$$

where user  $i \in V$  observes the new source  $Z'_i$  instead of  $Z_i$ . With such an increment to the private sources, the bit  $X_b$  can also be used as a secret key in addition to  $X_a$ , i.e., the secret key can be chosen as  $K = (X_a, X_b)$ . To achieve this, user 2 can reveal in public the XOR  $F := X_b \oplus X_c$ , and so user 3 can recover  $X_b$  by subtracting  $X_c$  from the sum. It can be shown that  $K$  is independent of  $F$  and is therefore kept secret from a wiretapper observing the entire public discussion. *The secrecy capacity is now equal to 2 bits.*

In the above, the addition of the private common randomness  $X_c$  increased the secrecy capacity by 1 bit. If we are allowed to choose who to give this common randomness to, the current choice of users 2 and 3 is in fact the most efficient (besides the equivalent choice of users 1 and 3). For example, if  $X_c$  were given to users 1 and 2 instead, then it is evident that the capacity would not have increased. Of course, one may choose to give  $X_c$  to user 1 in addition to users 2 and 3, where the capacity would be 2 bits. However, such an allocation is not considered efficient since additional resources, e.g., private communication, may be needed to give  $X_c$  to user 1.

For DSKA, we consider removing some common randomness from a subset of users, while trying not to diminish the secrecy capacity. It is easy to see that removing  $X_b$  from users 1 and 2 does not diminish the capacity, while removing  $X_a$  from all the users does.<sup>2</sup> In other words, the common randomness  $X_b$  is redundant but  $X_a$  is not. We can therefore consider the following simpler source for the purpose of achieving the secrecy capacity of 1 bit:

$$Z''_i := X_a \quad \text{for } i \in \{1, 2, 3\}.$$

Simplifying the source is useful because it simplifies the capacity-achieving scheme in [1] by reducing the amount of discussion required for the *communication for omniscience*.

### III. PROBLEM FORMULATION

We will formulate ISKA and DSKA as extensions of the secret key agreement problem in [1] under the source model without wiretapper's side information nor helpers. Readers may refer to [1] for the detailed secret key agreement protocol. In our formulation, we will only need the characterization (1.2a) [3] of the secrecy capacity. In this paper, we primarily denote sets with capital letters, random variables

<sup>2</sup>Note that we remove a common randomness from all the users observing it. The source is simpler after such a removal since its joint entropy reduces strictly by the amount of common randomness we remove. It is possible to extend this to more general multi-letter preprocessing of the random source, and doing so is potentially more useful for non-hypergraphical sources.

with the sans-serif font, and families of sets with script typeface letters. Furthermore, for a family  $\mathcal{F}$  of sets, we will use  $\text{minimal } \mathcal{F}$  and  $\text{maximal } \mathcal{F}$  to denote, respectively, the sets of inclusion-wise minimal and maximal elements of  $\mathcal{F}$ .

#### A. Incremental secret key agreement

To formulate ISKA, we consider adding a common randomness  $X$  of entropy  $\epsilon > 0$  to a subset  $S$  of users:

**Definition 3.1** For  $S \subseteq V$  and  $\epsilon > 0$ , we say that  $Z_V^{S, \epsilon}$  is an  $(S, \epsilon)$ -incremented source of  $Z_V$  if it can be written as

$$Z_i^{S, \epsilon} := \begin{cases} (Z_i, X) & i \in S \\ Z_i & \text{otherwise,} \end{cases} \quad (3.1)$$

where  $X$  is independent of  $Z_V$  and has entropy  $H(X) = \epsilon$ .  $\square$

We want to characterize the *rate of increase* in the secrecy capacity (1.2a) of the incremented source:

**Definition 3.2** The subderivative (one-sided limit)

$$\rho_S^+(Z_V) := \left. \frac{\partial I(Z_V^{S, \epsilon})}{\partial \epsilon} \right|_{\epsilon=0+} = \lim_{\epsilon \downarrow 0} \frac{I(Z_V^{S, \epsilon}) - I(Z_V)}{\epsilon} \quad (3.2)$$

is the *growth rate* of the secrecy capacity for the private source  $Z_V$  with respect to an infinitesimal increment in the common randomness of the subset  $S$ . Since the additional common randomness  $X$  is a valuable resource, we want to maximize the growth rate among all subsets  $S$  of the same size as in

$$\rho_k^+(Z_V) := \max_{S \subseteq V: |S| \leq k} \rho_S^+(Z_V) \quad (3.3)$$

for integer  $k$ . We refer to  $\rho_k^+$  as the *growth rate of order  $k$* .  $\square$

For an efficient allocation of common randomness, we want to identify small subsets with strictly positive growth rate:

**Definition 3.3**  $S \subseteq V$  is said to be a *critical (hyper)edge* if it is inclusion-wise *minimal* with  $\rho_S^+(Z_V) > 0$ . We use

$$\mathcal{S}_{\text{crit}}(Z_V) := \text{minimal}\{S \subseteq V \mid \rho_S^+(Z_V) > 0\} \quad (3.4)$$

to denote the set of all critical edges.<sup>3</sup>  $\square$

(We remark that a critical edge  $S$  is an edge in the  $(S, \epsilon)$ -incremented source.) Out of all the critical edges, the subsets with minimum size require the least resource. It is easy to argue that the minimum critical edges are the optimal solutions to (3.3) for the smallest  $k$  such that  $\rho_k^+(Z_V) > 0$ .

#### B. Decremental secret key agreement

To formulate the DSKA problem, we will consider a special kind of random sources:

<sup>3</sup>Unlike [14], the word critical is associated with an edge rather than a set family for a non-asymptotic hypergraphical source. Furthermore, the notion of critical family in [14] is related to the notion of excess edge in DSKA rather than the notion of critical edge in the ISKA problem. We also consider an asymptotic source model that is not restricted to be hypergraphical.

**Definition 3.4** We say that a source  $Z_V$  has an edge  $S \subseteq V$  if there is a common randomness  $X'$  that is observed only by the users in  $S$ , i.e., if we can rewrite  $Z_i$  (up to bijection) as

$$Z_i = \begin{cases} (Z'_i, X') & i \in S \\ Z'_i & \text{otherwise,} \end{cases} \quad (3.5)$$

where  $Z'_V$  is independent of  $X'$ . We reduce such a source to the following  $(S, \epsilon)$ -decremented source by removing an  $\epsilon \in (0, H(X'))$  amount of common randomness  $X'$ :

$$Z_{i,-\epsilon}^{S,-\epsilon} = \begin{cases} (Z'_i, X) & i \in S \\ Z'_i & \end{cases} \quad (3.6)$$

for some common randomness  $X$  independent of  $Z'_V$  and with  $H(X) = H(X') - \epsilon$ .  $\square$

Contrary to ISKA, we are interested in the *rate of decrease* in the secrecy capacity (1.2a):

**Definition 3.5** The subderivative (one-sided limit)

$$\rho_S^-(Z_V) := -\frac{\partial I(Z_V^{S,-\epsilon})}{\partial \epsilon} \Big|_{\epsilon=0^+} = \lim_{\epsilon \downarrow 0} \frac{I(Z_V) - I(Z_V^{S,-\epsilon})}{\epsilon}, \quad (3.7)$$

is the *loss rate* of the secrecy capacity for the private source  $Z_V$  with edge  $S$ .  $\square$

Unlike ISKA, we are interested in edges  $S$  with zero loss rate.

**Definition 3.6**  $S \subseteq V$  is said to be an *excess* or *redundant* edge if the corresponding loss rate is 0 for the source  $Z_V$  with edge  $S$ . In this case, we can simplify the secret key agreement by removing common randomness of the edge  $S$  without diminishing the secrecy capacity.  $\square$

#### IV. MAIN RESULTS

As pointed out by [6, 18] and elaborated in [4], the MMI can be computed in polynomial time using *submodular function minimization* [19] algorithms. A polynomial time algorithm was also given by an earlier work of Fujishige [20, 21] for a more general type of *submodular functions* and *set family*. It turns out that the characterization and computation of the growth rate, loss rate, critical edges, and excess edges depend only on the optimal partitions that attain the MMI (1.2a). The set of the optimal partitions will be denoted by

$$\Pi^*(Z_V) := \{P \in \Pi'(V) \mid I_P(Z_V) = I(Z_V)\}. \quad (4.1)$$

Using the combinatorial result of [22], the set  $\Pi^*(Z_V) \cup \{V\}$  forms a lattice, and hence admits a unique finest partitionm

$$\mathcal{P}^*(Z_V) := \min \Pi^*(Z_V). \quad (4.2)$$

Here, the minimum is with respect to the partial order “ $\prec$ ” of the partitions, which is defined as  $\mathcal{P} \prec \mathcal{P}'$  if the partition  $\mathcal{P}$  is *finer than* (or a *refinement of*) the partition  $\mathcal{P}'$ . In other words,  $\mathcal{P}$  can be obtained by further partitioning one or more subsets in  $\mathcal{P}'$ . Following [4], we will refer to the finest partition  $\mathcal{P}^*(Z_V)$  as the *fundamental partition*. This partition has an elegant interpretation [12] in data clustering, and furthermore, can be computed in strongly polynomial time using algorithms such as [23] applied to the *minimum average cost clustering*.

For ISKA, we can characterize the growth rate and critical edges using the optimal partitions as follows:

**Theorem 4.1** For any  $Z_V$  and  $S \subseteq V$ ,

$$\rho_S^+(Z_V) = \min_{P \in \Pi^*(Z_V)} \frac{\sum_{C \in P} \chi_{\{C \cap S \neq \emptyset\}} - \chi_{\{S \neq \emptyset\}}}{|P| - 1}, \quad (4.3a)$$

where  $\chi$  is the indicator function of the condition specified in the subscript. It follows that

$$\mathcal{S}_{\text{crit}}(Z_V) = \text{minimal}\{S \subseteq V \mid S \not\subseteq C, \forall C \in \mathcal{P} \in \Pi^*(Z_V)\}. \quad (4.3b)$$

In other words,  $S \subseteq V$  is critical iff it is a minimal set that overlaps at least two blocks of every optimal partition.  $\square$

PROOF See Appendix A.  $\blacksquare$

Indeed,  $\mathcal{S}_{\text{crit}}(Z_V)$  depends on  $Z_V$  only through the coarsest optimal partitions in  $\Pi^*(Z_V)$ . This is because, if  $S$  crosses a partition, i.e., overlaps at least two blocks of the partition, then it must also cross any refinement of the partition.

For DSKA, we can similarly characterize the loss rate and excess edges as follows:

**Theorem 4.2** For any  $Z_V$  with edge  $S \subseteq V$ ,

$$\rho_S^-(Z_V) = \max_{P \in \Pi^*(Z_V)} \frac{\sum_{C \in P} \chi_{\{C \cap S \neq \emptyset\}} - \chi_{\{S \neq \emptyset\}}}{|P| - 1}. \quad (4.4a)$$

It follows that  $S$  is an excess edge iff

$$\exists C \in \mathcal{P}^*(Z_V), S \subseteq C, \quad (4.4b)$$

i.e.,  $S$  does not cross the fundamental partition.  $\square$

PROOF See Appendix A.  $\blacksquare$

The condition for an excess edge depends only on the fundamental partition, and therefore can be checked in strongly polynomial time.

Equations (4.3a) and (4.3b) imply the following simple properties of  $\rho_k^+$  (3.3) and  $\mathcal{S}_{\text{crit}}$ .

**Proposition 4.1**  $\rho_k^+(Z_V)$  is non-decreasing in  $k$  and equal to 0 for  $k \leq 1$ . Furthermore,

$$\rho_k^+(Z_V) = 1 \quad \text{iff} \quad k \geq |\mathcal{P}^*(Z_V)|,$$

the size of the fundamental partition. Finally, we have at least one critical edge, i.e.  $\mathcal{S}_{\text{crit}}(Z_V) \neq \emptyset$ , and the minimum size of a critical edge is at least 2, i.e.,  $\min\{|S| \mid S \in \mathcal{S}_{\text{crit}}(Z_V)\} \geq 2$ .  $\square$

PROOF See Appendix A.  $\blacksquare$

Computing  $\rho_k^+$  in general can be quite difficult but some simple cases will be given in the next section. Surprisingly, it turns out that computing a minimum critical edge can be done in strongly polynomial time, which is due to the result below.

**Theorem 4.3** All critical edges in  $\mathcal{S}_{\text{crit}}(Z_V)$  have the same size, and are therefore minimum.  $\square$

In other words, all the critical edges are minimum. A critical edge can be obtained easily as follows: Starting with  $S = V$ , repeatedly remove an element from  $S$  as long as doing so does

not violate  $I(Z_V^{S,1}) > I(Z_V)$ . The condition can be checked in strongly polynomial time for at most  $O(|V|^2)$  times.

Indeed, a stronger result can be proved. We will characterize the critical edges completely using only the maximal blocks from the optimal partitions,

$$\mathcal{T}_{\max}(Z_V) := \text{maximal} \bigcup \Pi^*(Z_V) \quad (4.5)$$

which can also be computed in strongly polynomial time. More precisely, we can show that:

**Lemma 4.1** *Either one of the following cases happen:*

$$\mathcal{T}_{\max}(Z_V) \in \Pi^*(Z_V) \quad (4.6)$$

$$\bar{\mathcal{T}}_{\max}(Z_V) := \{V \setminus C : C \in \mathcal{T}_{\max}(Z_V)\} \in \Pi'(U) \quad (4.7)$$

for some  $U \subseteq V$ . In words, either  $\mathcal{T}_{\max}(Z_V)$  is an optimal partition or its complement  $\bar{\mathcal{T}}_{\max}(Z_V)$  is a set of at least two non-empty disjoint subsets of  $V$ . Indeed, (4.6) means that  $\mathcal{T}_{\max}(Z_V)$  is the unique coarsest optimal partition.  $\square$

**Theorem 4.4** *If (4.6) happens,*

$$\mathcal{S}_{\text{crit}}(Z_V) = \{\{i, j\} \mid i \in C, j \in V \setminus C, C \in \mathcal{T}_{\max}(Z_V)\} \quad (4.8)$$

and so all the critical edges have size 2. If (4.7) happens,

$$\mathcal{S}_{\text{crit}}(Z_V) = \{\{i_C \mid C \in \mathcal{T}_{\max}(Z_V)\} \mid i_C \in V \setminus C\}, \quad (4.9)$$

which is taken to mean the collection of the sets  $\{i_C \mid C \in \mathcal{T}_{\max}(Z_V)\}$  of representatives  $i_C$  of subsets  $V \setminus C$  for  $C \in \mathcal{T}_{\max}(Z_V)$ . It follows that  $|\mathcal{T}_{\max}(Z_V)|$  is the size of all the critical edges.  $\square$

Note that, Theorem 4.4 implies Theorem 4.3 immediately.

PROOF See Appendix B.  $\blacksquare$

**Example 4.1** Consider  $V = \{1, 2, 3\}$ . Let  $Z_1 = Z_2$  be a uniformly random bit, and  $Z_3 = 0$ . It can be shown that

$$\begin{aligned} \Pi^*(Z_V) &= \{\{\{1, 2\}, \{3\}\}\} \\ \mathcal{S}_{\text{crit}}(Z_V) &= \{\{1, 3\}, \{2, 3\}\}. \end{aligned}$$

There is a unique optimal partition and so  $\mathcal{T}_{\max}(Z_V) = \mathcal{P}^*(Z_V) = \{\{\{1, 2\}, \{3\}\} \in \Pi^*(Z_V)$ , satisfying (4.6). The set of critical edges by (4.8) is  $\mathcal{S}_{\text{crit}}(Z_V) = \{\{1, 3\}, \{2, 3\}\}$ .

Indeed, (4.6) may hold even when the optimal partition is not unique. For instance, consider  $V = \{1, 2, 3\}$  and let  $Z_1 := (X_a, X_b, X_c)$ ,  $Z_2 := (X_a, X_b, X_d)$  and  $Z_3 := (X_c, X_d)$  where  $X_i$ 's are uniformly random and independent bits. It follows that

$$\begin{aligned} \Pi^*(Z_V) &= \{\{\{1, 2\}, \{3\}\}, \{\{1\}, \{2\}, \{3\}\}\} \\ \mathcal{T}_{\max}(Z_V) &= \{\{\{1, 2\}, \{3\}\} \in \Pi^*(Z_V) \\ \mathcal{S}_{\text{crit}}(Z_V) &= \{\{1, 3\}, \{2, 3\}\} \end{aligned}$$

which satisfies (4.6) but  $\mathcal{T}_{\max}$  is the coarsest partition rather than the fundamental partition.  $\square$

**Example 4.2** Let  $V = \{1, 2, 3, 4\}$ , and

$$Z_1 = X_a, \quad Z_2 = (X_a, X_b), \quad Z_3 = (X_b, X_c), \quad \text{and} \quad Z_4 = X_c$$

where  $X_i$ 's are independent uniformly random bits. The optimal partitions in  $\Pi^*(Z_V)$  are shown in Fig. 1b, and  $\mathcal{T}_{\max}(Z_V) = \{\{1, 2, 3\}, \{2, 3, 4\}\}$ . Although  $\mathcal{T}_{\max}(Z_V)$  is

not an optimal partition,  $\bar{\mathcal{T}}_{\max}(Z_V) = \{\{4\}, \{1\}\}$ , which satisfies (4.7). By (4.9), the set of minimum critical edges is  $\mathcal{S}_{\text{crit}}(Z_V) = \{\{1, 4\}\}$ . It crosses at least two blocks of every coarsest optimal partition, and therefore every optimal partition.

Example 4.2 is a special case of the pairwise independent network (PIN) source model [24], where in this example the source is simply a tree, see Fig. 1a. For tree PINs in general, it can be argued that a partition  $\mathcal{P} \in \Pi'(V)$  is optimal iff each block  $C \in \mathcal{P}$  induces a tree on  $C$ . For example, the optimal partition  $\{\{1, 2\}, \{3, 4\}\}$  in Fig. 1b induces two subtrees, one connecting 1 and 2, and the other connecting 3 and 4. Thus, the fundamental partition is always the partition into singletons, but it is not the only optimal partition for  $|V| \geq 3$ . Nevertheless, we can show that there is only one critical edge and such an edge is equal to the set of leaves.  $\square$

## V. COMPUTING THE GROWTH RATE OF DIFFERENT ORDERS

In this section, we will illustrate the computation of  $\rho_k^+$  in (3.3). There are simple cases where  $\rho_k^+$  (and therefore  $\mathcal{S}_{\text{crit}}$ ) can be computed easily. For example, in the special case when the fundamental partition is the unique optimal partition, i.e.  $|\Pi^*(Z_V)| = 1$ , it can be argued easily from (4.3a) that

$$\rho_k^+(Z_V) = \frac{k-1}{|\mathcal{P}^*(Z_V)|-1} \quad \text{for } k \leq |\mathcal{P}^*(Z_V)|,$$

where an optimal solution  $S$  to (3.3) is any set of  $k$  elements, each from a different block of the fundamental partition. In particular,

$$\mathcal{S}_{\text{crit}}(Z_V) = \{\{i, j\} \mid i \in C, j \in V \setminus C, C \in \mathcal{P}^*(Z_V)\}$$

and so all the critical edges are minimum with size 2. When the partition into singletons is the unique optimal partition, then the optimal solution  $S$  to (3.3) is simply any  $k$ -subset of  $V$ . In particular, the singleton partition is shown to be the unique optimal partition in [7] for any PIN model that corresponds to a complete graph. We can also show the same result for any PIN model that corresponds to a cycle. In general, it is possible to check in strongly polynomial time whether the fundamental partition is the unique optimal partition. (See Appendix C.) For instance, in Example 4.1, the optimal partition is unique and all the critical edges have size 2 as expected. More details about the computation and interpretations of the fundamental partition can be found in [4].

When the optimal partition is not unique,  $\rho_k^+(Z_V)$  may not be linear in  $k$ . The marginal increase in growth rate may not be diminishing in  $k$  either. This is the case, for instance, for Example 4.2 as shown in Fig. 1c.

If  $H(Z_B)$  is an integer for every  $B \subseteq V$ , then  $\rho_S^+(Z_V)$  can be computed in strongly polynomial time. To argue this, choose  $\epsilon = \frac{1}{|V|}$ . Note that  $I_{\mathcal{P}}(Z_V)/\epsilon$  is an integer and so  $I_{\mathcal{P}}(Z_V)$  for different  $\mathcal{P} \in \Pi'(V) \setminus \Pi^*(Z_V)$  is larger than  $I(Z_V)$  by at least  $\epsilon$ , while  $I(Z_V^{S,\epsilon})$  is larger than  $I(Z_V)$  by at most  $\epsilon$ . Thus,  $\Pi^*(Z_V^{S,\epsilon}) \subseteq \Pi^*(Z_V)$  and so  $\rho_S^+(Z_V) = \frac{I(Z_V^{S,\epsilon}) - I(Z_V)}{\epsilon}$ . Each term on the R.H.S. can be computed in strongly polynomial time. A more refined argument suggests that one can choose any  $\epsilon \leq \frac{1}{(|V|-1)(|V|-2)}$ .

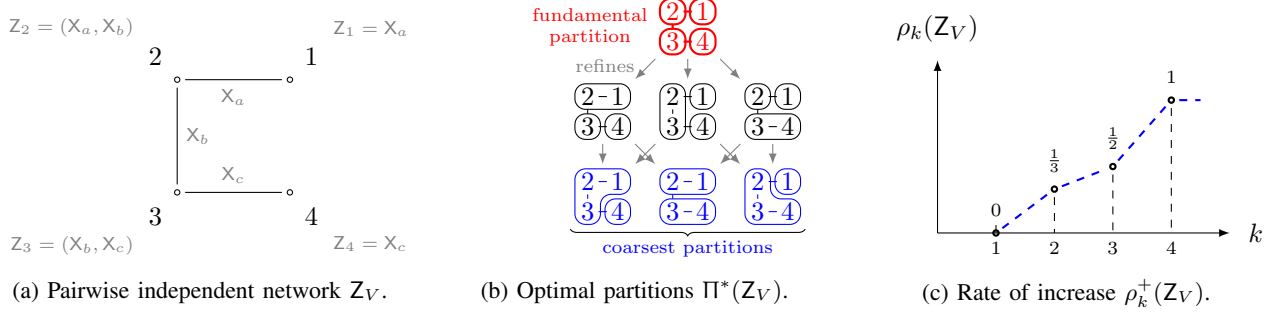


Fig. 1: Optimal partitions and rate of increase of the tree network in Example 4.2.

## VI. CONCLUSION

In this work, we have formulated the ISKA and DSKA problems to study how the MMI of a set of random variables changes with respect to an infinitesimal increment and decrement in the MMI of a subset of the random variables. We found that the set of optimal partitions that attain the MMI of a set of random variables completely characterizes the growth rate for ISKA and the loss rate for DSKA.

For ISKA, the growth rate can be computed easily in some special cases, e.g., when the optimal partition is unique. In general, however, it is not clear whether the computation can be done in polynomial time. The growth rate can be non-linear in the order, and the marginal return may even increase as we increase the order. Very surprisingly, however, a minimum critical edge can be computed in strongly polynomial time because all critical edges have the same size. In other words, one can easily identify a minimum subset of users to give an additional common randomness to, such that the secrecy capacity strictly increases. For DSKA, the condition for an edge to be redundant can be characterized in strongly polynomial time using the fundamental partition. Identifying excess edges is useful in simplifying secret key agreement schemes. In particular, it is hopeful that further investigation can resolve the conjectures regarding the communication complexity for secret key agreement [15–17].

## APPENDIX A

### PROOFS OF BASIC PROPERTIES

PROOF (THEOREM 4.1) Clearly,  $\rho_\emptyset^+(Z_V) = 0$ . Consider  $S \subseteq V : S \neq \emptyset$ . Rewriting the divergence in terms of the entropy as  $D(P_{Z_V} \| \prod_{C \in \mathcal{P}} P_{Z_C}) = \sum_{C \in \mathcal{P}} H(Z_C) - H(Z_V)$ , we have

$$\begin{aligned} I(Z_V^{\epsilon, S}) &= \min_{\mathcal{P} \in \Pi'(V)} \frac{1}{|\mathcal{P}| - 1} \left[ \sum_{C \in \mathcal{P}} H(Z_C^{\epsilon, S}) - H(Z_V^{\epsilon, S}) \right] \\ &= \min_{\mathcal{P} \in \Pi'(V)} \frac{1}{|\mathcal{P}| - 1} \left[ \sum_{C \in \mathcal{P}} H(Z_C) - H(Z_V) \right. \\ &\quad \left. + \sum_{C \in \mathcal{P}} H(X) \chi_{\{S \cap C \neq \emptyset\}} - H(X) \right] \\ &= \min_{\mathcal{P} \in \Pi'(V)} \left[ I_{\mathcal{P}}(Z_V) + \epsilon \frac{\sum_{C \in \mathcal{P}} \chi_{\{C \cap S \neq \emptyset\}} - 1}{|\mathcal{P}| - 1} \right]. \end{aligned}$$

Suppose  $I(Z_V^{\epsilon, S}) < \min_{\mathcal{P} \in \Pi'(V) \setminus \Pi^*(Z_V)} I_{\mathcal{P}}(Z_V)$ , which is possible for all  $\epsilon > 0$  sufficiently small since  $|\Pi'(V)|$  is finite.

It does not lose optimality to restrict  $\mathcal{P}$  to  $\Pi^*(Z_V)$  and so we have for all  $\epsilon > 0$  sufficiently small that

$$I(Z_V^{\epsilon, S}) = I(Z_V) + \epsilon \min_{\mathcal{P} \in \Pi^*(Z_V)} \frac{\sum_{C \in \mathcal{P}} \chi_{\{C \cap S \neq \emptyset\}} - 1}{|\mathcal{P}| - 1}$$

which gives (4.3a). (4.3b) follows from (4.3a) directly. ■

PROOF (THEOREM 4.2) The proof of (4.4a) is analogous to the proof of (4.3a) above, but we have max instead of min due to a sign change because the rate is on the loss rather than growth of the MMI. To derive the condition (4.4b) for excess edge, notice that (4.4a) is zero iff  $S$  does not cross any optimal partitions. It suffices to consider only the finest optimal partition, namely  $\mathcal{P}^*(Z_V)$ , because  $S$  does not cross  $\mathcal{P}^*(Z_V)$  implies it does not cross any coarser optimal partitions, which cover all the optimal partitions. ■

PROOF (PROPOSITION 4.1)  $\rho_k^+(Z_V)$  is non-decreasing in  $k$  because  $\rho_S^+(Z_V)$  by (4.3a) is non-decreasing in  $S$  with respect to set inclusion. It is equal to 0 for  $k = 1$  because  $\rho_{\{i\}}^+(Z_V) = 0$  for all  $i \in V$ . This also means that a critical edge, if any, must be non-singleton, with size at least two.  $\rho_k^+(Z_V)$  is at most 1 because  $\rho_V^+(Z_V) = 1$ . More precisely,  $\rho_S^+(Z_V) = 1$  iff  $S \cap C \neq \emptyset$  for every  $C \in \mathcal{P}^*$  and  $\mathcal{P}^* \in \Pi^*(Z_V)$ . Since the fundamental partition  $\mathcal{P}^*(Z_V)$  is the unique finest optimal partition, we have  $\rho_S^+(Z_V) = 1$  iff  $S \cap C \neq \emptyset$  for every  $C \in \mathcal{P}^*(Z_V)$ . That means  $\rho_S^+(Z_V) < 1$  if  $|S| < \text{abs} \mathcal{P}^*(Z_V)$ , and  $\rho_S^+(Z_V) = 1$  for any  $S$  obtained by taking at least one element from each block in the fundamental partition. The fact that  $\rho_V^+(Z_V) = 1$  also means that there is at least one critical edge. ■

## APPENDIX B

### PROOF OF THEOREM 4.4

PROOF (THEOREM 4.4) From (4.3b), we have  $S \in \mathcal{S}_{\text{crit}}(Z_V)$  iff

$$S \not\subseteq C \quad \text{or equiv.} \quad S \setminus C \neq \emptyset \quad \forall C \in \mathcal{T}_{\text{max}}(Z_V). \quad (\text{B.1})$$

From this, it can be argued easily that the sets defined in (4.8) and (4.9) are critical edges for the cases (4.6) and (4.7) respectively. If  $S$  is a critical edge under (4.6), any element, say  $i \in S$ , must be contained by some  $C \in \mathcal{T}_{\text{max}}(Z_V)$  since  $\mathcal{T}_{\text{max}}(Z_V)$  is a partition of  $V$ . By (B.1), it must contain an element  $j \in V \setminus C$  and so  $S$  must be in (4.8) as desired. If  $S$  is a critical edge under (4.7), it must contain an element from  $V \setminus C$  for every  $C \in \mathcal{T}_{\text{max}}(Z_V)$  by (B.1). Thus, it must be in (4.9) as desired. ■

It remains to prove Lemma 4.1. We do so using the idea of *zero-singleton-submodular* function in [22]. Denote the fundamental partition as

$$\mathcal{P}^*(Z_V) = (C_1^*, \dots, C_\ell^*) \quad (\text{B.2})$$

by indexing the blocks from 1 to  $\ell = |\mathcal{P}^*(Z_V)|$ . Define  $g : 2^{[\ell]} \mapsto \mathbb{R}$  as

$$g(B) := h_\gamma \left( \bigcup_{i \in B} C_i^* \right) - \sum_{i \in B} h_\gamma(C_i^*) \quad \text{for } B \subseteq [\ell] \quad (\text{B.3})$$

with  $\gamma := I(RZ_V)$  and  $h_\gamma(C) := H(Z_C) - \gamma$  is the residual randomness defined in [4]. It follows immediately that  $g$  is submodular with

$$g(\{i\}) = 0 \quad \text{for } i \in [\ell], \quad (\text{B.4})$$

and is therefore called a zero-singleton-submodular function. It can also be shown to be non-negative over non-empty sets.

**Proposition B.1 ([22, p.198])**  $g(B) \geq 0$  for all  $B \subseteq [\ell] : B \neq \emptyset$ . The zero sets of  $g$ ,

$$\mathcal{Z}(g) := \{B \subseteq [\ell] : g(B) = 0\} \quad (\text{B.5})$$

forms an intersecting family, i.e.

$$U \cap W, U \cup W \in \mathcal{Z}(g) \text{ for all } U, W \in \mathcal{Z}(g) : U \cap W \neq \emptyset \quad (\text{B.6})$$

Furthermore,

$$\left\{ \bigcup_{i \in B} C_i^* : B \in \mathcal{Z}(g) \right\} = \bigcup \Pi^*(Z_V) \cup \{V\}, \quad (\text{B.7})$$

and so  $\bigcup \Pi^*(Z_V) \cup \{V\}$  is also an intersecting family.  $\square$

**PROOF** For any partition  $\mathcal{P} \in \Pi([\ell])$ ,

$$\begin{aligned} g[\mathcal{P}] &:= \sum_{C \in \mathcal{P}} g(C) \\ &= \sum_{C \in \mathcal{P}} h_\gamma \left( \bigcup_{i \in C} C_i^* \right) - \sum_{i=1}^{\ell} h_\gamma(C_i^*) \\ &= h_\gamma \left[ \left\{ \bigcup_{i \in C} C_i^* : C \in \mathcal{P} \right\} \right] - h_\gamma[\mathcal{P}^*(Z_V)] \geq 0, \end{aligned}$$

because, by [4, Theorem 5.1],  $\mathcal{P}^*(Z_V)$  minimizes  $h_\gamma$  over all partitions of  $V$ , which include  $\{\bigcup_{i \in C} C_i^* : C \in \mathcal{P}\}$ . Furthermore, we have equality  $g[\mathcal{P}] = 0$  for  $\mathcal{P} \in \Pi([\ell])$  iff

$$\left\{ \bigcup_{i \in C} C_i^* : C \in \mathcal{P} \right\} \in \Pi^*(Z_V).$$

Suppose to the contrary that  $g(B) < 0$  for some non-empty  $B \subseteq [\ell]$ . Then,  $g(\{B\} \cup \{\{i\}, i \in [\ell] \setminus B\}) < 0$  by the zero-singleton property (B.4), but this contradicts  $g[\mathcal{P}] \geq 0$  above for all  $\mathcal{P} \in \Pi([\ell])$ . This proves the non-negativity of  $g$  over non-empty sets.

For  $U, W \in \mathcal{Z}(g) : U \cap W \neq \emptyset$ , we have

$$0 \geq g(U) - g(U \cap W) \geq g(U \cup W) - g(W) \geq 0$$

where the second inequality is by the submodularity of  $g$ , and the first and last inequalities are because  $g(U) = g(W) = 0$  and  $g(U \cap W), g(U \cup W) \geq 0$  by the non-negativity of  $g$  over non-empty sets argued above. Thus, all inequalities are

satisfied with equality and so  $g(U \cap W) = g(U \cup W) = 0$  as desired for  $\mathcal{Z}(g)$  to be an intersecting family.

It remains to prove (B.7). By [4, Theorem 5.2], every partition in  $\Pi^*(Z_V) \cup \{\{V\}\}$  is coarser than  $\mathcal{P}^*(Z_V)$  and can therefore be expressed as  $\{\bigcup_{i \in C} C_i^* : C \in \mathcal{P}\} \in \Pi'(V)$  for some  $\mathcal{P} \in \Pi'([\ell])$ . By optimality,  $g[\mathcal{P}] = 0$ , and so  $g(C) = 0$  for all  $C \in \mathcal{P}$  since  $C$  is non-empty and  $g$  is non-negative over non-empty sets as argued before. Conversely, suppose  $g(B) = 0$ . Then  $g(\{B\} \cup \{\{i\}, i \in [\ell] \setminus B\}) = 0$  by (B.4), and so  $\bigcup_{i \in B} C_i^*$  is a block of an optimal partition, namely  $\{\bigcup_{i \in B} C_i^* \} \cup \{C_i^* : i \in [\ell] \setminus B\}$ . This completes the proof of (B.7). Since  $C_i^*$ 's are disjoint, the fact that  $\mathcal{Z}(g)$  is an intersecting family implies that  $\bigcup \Pi^*(Z_V) \cup \{V\}$  is.  $\blacksquare$

**PROOF (LEMMA 4.1)** We first argue that: for any distinct  $C_1, C_2 \in \mathcal{T}_{\max}(Z_V)$ , we have

$$C_1 \cap C_2 \neq \emptyset \text{ implies } C_1 \cup C_2 = V. \quad (\text{B.8})$$

To show this, note that  $C_1$  and  $C_2$  are maximal sets from  $\bigcup \Pi^*(Z_V)$  by definition (4.5). By Proposition B.1,  $\bigcup \Pi^*(Z_V) \cup \{V\}$  is an intersecting family and so  $C_1 \cap C_2 \neq \emptyset$  implies  $C_1 \cup C_2$  is also in the family. Suppose to the contrary that  $C_1 \cup C_2 \neq V$ , then  $C_1 \cup C_2$  is a strictly larger set in  $\bigcup \Pi^*(Z_V)$  than the distinct sets  $C_1$  and  $C_2$ , which contradicts the maximality.

Next, we argue that if there exists distinct  $C_1, C_2 \in \mathcal{T}_{\max}(Z_V)$  such that  $C_1 \cap C_2 \neq \emptyset$ , then

$$C \cap C' \neq \emptyset \text{ for all } C, C' \in \mathcal{T}_{\max}(Z_V). \quad (\text{B.9})$$

Indeed, we have a stronger statement that

$$\emptyset \neq (V \setminus C_1) \subseteq C \text{ for all } C \in \mathcal{T}_{\max}(Z_V) \setminus \{C_1\}. \quad (\text{B.10})$$

If  $C = C_2$ , we have (B.10) directly from (B.8). Suppose  $C \neq C_2$ . Then,  $C_1 \cap C \neq \emptyset$  because  $C \cap V \setminus C_2 \neq \emptyset$  by the maximality of  $C \neq C_2$ , and  $V \setminus C_2 \subseteq C_1$  by (B.8) that  $C_1 \cup C_2 = V$ . Applying (B.8) again with  $C_2$  replaced by  $C$ , we have  $C_1 \cup C = V$  as desired.

By the contrapositive statement of (B.10), if  $C_1 \cap C_2 = \emptyset$  for some distinct  $C_1, C_2 \in \mathcal{T}_{\max}(Z_V)$ , then all the sets in  $\mathcal{T}_{\max}(Z_V)$  must be disjoint, and so we have (4.6) since every element in  $V$  must be covered by at least one block of an optimal partition. In the other case when every distinct  $C_1, C_2 \in \mathcal{T}_{\max}(Z_V)$  must intersect, (B.8) implies  $C_1 \cup C_2 = V$ , or equivalently,  $(V \setminus C_1) \cap (V \setminus C_2) = \emptyset$ , which implies (4.7) as desired.  $\blacksquare$

By (B.7),  $\mathcal{T}_{\max}(Z_V)$  can be obtained from the maximal zero sets in  $\mathcal{Z}(g)$  together with the fundamental partitions  $\mathcal{P}^*(Z_V)$ . The maximal zero set not containing an element  $i \in [\ell]$ , can be computed in strongly polynomial time as follows: Starting with  $C = \emptyset$ , add an element from  $[\ell] \setminus \{i\}$  to  $C$  repeatedly as long as  $\min_{C' \in 2^{[\ell] \setminus \{i\}} : C \subseteq C'} g(C') = 0$ . The minimization is repeated at most  $O(|V|^2)$  times and can be solved in strongly polynomial time using some existing algorithm for submodular function minimization over a lattice family [19]. Hence,  $\mathcal{T}_{\max}(Z_V)$  and therefore  $\mathcal{S}_{\text{crit}}(Z_V)$  can be computed in strongly polynomial time by Theorem 4.4.



### Conjecture:

Does critical edges have the same rate of increase? We conjecture that  $\rho_S^+ = \frac{|S|-1}{|\mathcal{P}^*(Z_V)|-1}$  for all  $S \in \mathcal{S}_{\text{crit}}(Z_V)$ .

## APPENDIX C

### UNIQUENESS OF THE OPTIMAL PARTITION

To check whether there is a unique optimal partition in strongly polynomial time, we can make use of the zero-singleton-submodular function  $g$  defined in (B.3) based on the fundamental partition  $\mathcal{P}^*(Z_V)$  and the MMI. In essence of (B.7), we only need to check whether the zero sets  $Z(g)$  in (B.5) consists only of the singletons. This can be computed in  $O(|V|^2)$  submodular function minimizations, namely  $\min_{B \supseteq \{i,j\}} g(B)$  for all pair  $(i, j)$  of distinct elements.  $\mathcal{P}^*(Z_V)$  and  $I(Z_V)$  can also be computed in  $O(|V|^2)$  submodular function minimizations [23].

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## REFERENCES

- [1] I. Csiszár and P. Narayan, "Secrecy capacities for multiple terminals," *IEEE Trans. Inf. Theory*, vol. 50, no. 12, Dec. 2004.
- [2] C. Chan, "On tightness of mutual dependence upperbound for secret-key capacity of multiple terminals," *arXiv preprint arXiv:0805.3200*, 2008.
- [3] C. Chan and L. Zheng, "Mutual dependence for secret key agreement," in *Proc. of 44th Annual Conf. Inf. Sciences and Systems*, 2010.
- [4] C. Chan, A. Al-Bashabsheh, J. Ebrahimi, T. Kaced, and T. Liu, "Multivariate mutual information inspired by secret-key agreement," *Proc. IEEE*, vol. 103, no. 10, pp. 1883–1913, Oct 2015.
- [5] S. Nitinawarat and P. Narayan, "Perfect omniscience, perfect secrecy, and steiner tree packing," *IEEE Trans. Inf. Theory*, vol. 56, no. 12, pp. 6490–6500, Dec. 2010.
- [6] C. Chan, "The hidden flow of information," in *Proc. IEEE Int. Symp. on Inf. Theory*, St. Petersburg, Russia, Jul. 2011.
- [7] M. Mukherjee, N. Kashyap, and Y. Sankarasubramaniam, "Achieving SK capacity in the source model: When must all terminals talk?" in *Proc. of IEEE Int. Symp. on Inf. Theory*, June 2014.
- [8] M. Mukherjee and N. Kashyap, "The communication complexity of achieving sk capacity in a class of pin models," in *Proc. of IEEE Int. Symp. on Inf. Theory*, June 2015, pp. 296–300.
- [9] H. Zhang, Y. Liang, and L. Lai, "Secret key capacity: Talk or keep silent?" in *Proc. IEEE Int. Symp. on Inf. Theory*, June 2015.
- [10] C. Chan, A. Al-Bashabsheh, Q. Zhou, N. Ding, T. Liu, and A. Sprintson, "Successive omniscience," *IEEE Trans. Inf. Theory*, vol. PP, no. 99, pp. 1–1, 2016.
- [11] N. Ding, R. Kennedy, and P. Sadeghi, "Iterative merging algorithm for cooperative data exchange," in *Network Coding (NetCod), 2015 International Symposium on*, June 2015, pp. 41–45.
- [12] C. Chan and T. Liu, "Clustering of random variables by multivariate mutual information on Chow-Liu tree approximations," in *Fifty-Third Annual Allerton Conference on Communication, Control, and Computing*, Allerton Retreat Center, Monticello, Illinois, Sep. 2015.
- [13] C. Chan, A. Al-Bashabsheh, Q. Zhou, T. Kaced, and T. Liu, "Info-clustering: A mathematical theory of clustering," submitted to *IEEE Trans. Mol. Biol. Multi-Scale Commun.* on Apr 30, 2016.
- [14] T. A. Courtade and T. R. Halford, "Coded cooperative data exchange for a secret key," *IEEE Trans. Inf. Theory*, vol. PP, no. 99, pp. 1–1, 2016.
- [15] M. Mukherjee, N. Kashyap, and Y. Sankarasubramaniam, "On the public communication needed to achieve sk capacity in the multiterminal source model," *IEEE Trans. Inf. Theory*, vol. PP, no. 99, pp. 1–1, 2016.
- [16] M. Mukherjee, C. Chan, N. Kashyap, and Q. Zhou, "Bounds on public communication needed to achieve sk capacity in the hypergraphical source model," submitted to *ISIT* 2016.
- [17] C. Chan, M. Mukherjee, N. Kashyap, and Q. Zhou, "When is omniscience a rate-optimal strategy for achieving secret key capacity?" submitted to *ITW* 2016.
- [18] N. Milosavljevic, S. Pawar, S. El Rouayheb, M. Gastpar, and K. Ramchandran, "Deterministic algorithm for the cooperative data exchange problem," in *Proc. of IEEE Int. Symp. on Inf. Theory*, Jul. 2011.
- [19] A. Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency*. Springer, 2002.
- [20] S. Fujishige, "Structures of polyhedra determined by submodular functions on crossing families," *Mathematical Programming*, vol. 29, no. 2, pp. 125–141, 1984.
- [21] —, "Optimization over the polyhedron determined by a submodular function on a co-intersecting family," *Mathematical Programming*, vol. 42, no. 1-3, pp. 565–577, 1988.
- [22] H. Narayanan, "The principal lattice of partitions of a submodular function," *Linear Algebra and its Applications*, vol. 144, no. 0, pp. 179 – 216, 1990.
- [23] K. Nagano, Y. Kawahara, and S. Iwata, "Minimum average cost clustering," in *NIPS*, J. D. Lafferty, C. K. I. Williams, J. Shawe-Taylor, R. S. Zemel, and A. Culotta, Eds. Curran Associates, Inc., 2010, pp. 1759–1767.
- [24] S. Nitinawarat, C. Ye, A. Barg, P. Narayan, and A. Reznik, "Secret key generation for a pairwise independent network model," *IEEE Trans. Inf. Theory*, vol. 56, no. 12, pp. 6482–6489, Dec 2010.



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